الأكاديمية العربية الدولية
Arab International Academy

إسم المادة: المعادلات التفاضلية ذات الرتب الكسرية

إسم الدكتور: الدكتورة جمانة حكمت الخالصي

الأكاديمية العربية الدولية – منصة أعد

Fractional Differential Equations

- What is Fractional Calculus
  - What are fractional differential eqautions
- Historical Review
- Definitions
- Applications
- Solving Fractional Differential Equations

What is Fractional Calculus ?

- All of us are familiar with *normal* derivatives and integrals, like, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}$, $\int_0^t f(u)du$.
- We have first-order, second-order derivatives, or first integral, double integral, of a function.
- Now we wish to have half-order, πth-order, or ...... (3-6i)th-order derivative of a function.
- So, Fractional calculus $\Rightarrow$ derivatives and integrals of arbitrary real, or complex order

What are Fractional Differential Equations?

- Fractional differential equations (FDEs) involve derivatives of fractional order.
- Fractional derivatives generalize the concept of integer-order derivatives to non-integer orders.
- FDEs have been increasingly studied in many scientific fields such as engineering, physics, biology, and finance.
- They capture the memory and hereditary properties of systems.

Historical Review

- The first idea raised in 1695 by Leibniz when he wrote a letter to L'Hospital where he said :

"Can the meaning of derivatives with integer order to be generalized to derivatives with non –integer orders?"

-To this L'Hospital replied with a question of his own: '***What if the order will be ½ ?***'

-To this, Leibniz said: '***It will lead to a paradox, form which one day useful consequences will be drawn.***'

-This letter of Leibniz was dated 30th September, 1695. So 30th September is considered as the birthday of fractional calculus.

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Definitions

- The **Riemann-Liouville** (RL) fractional integral of order $0 \leq \alpha \leq 1$ is defined as

$$J^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{(\alpha-1)} f(u) du, \quad t > 0,$$

- We can define fractional derivative of order α by two ways:
  1. **RL** fractional derivative: Take fractional integral of order $(1-\alpha)$ and then take a first derivative,

$$D_t^{\alpha} f(t) = \frac{d}{dt} J^{1-\alpha} f(t)$$

  2. **Caputo** fractional derivative: Take first order derivative and then take a fractional integral of order $(1-\alpha)$,

$$D_t^{\alpha} f(t) = J^{1-\alpha} \frac{d}{dt} f(t)$$

Definitions

- The most commonly used fractional derivative is the Caputo derivative. It is defined as the convolution of the function with a fractional order power function, where the order of the derivative is between 0 and 1. The Caputo derivative is a generalization of the classical derivative, and it has the property of producing non-local effects, which means that the value of the derivative at a certain point may depend on the values of the function at other points.

- Comparison with integer-order derivatives.

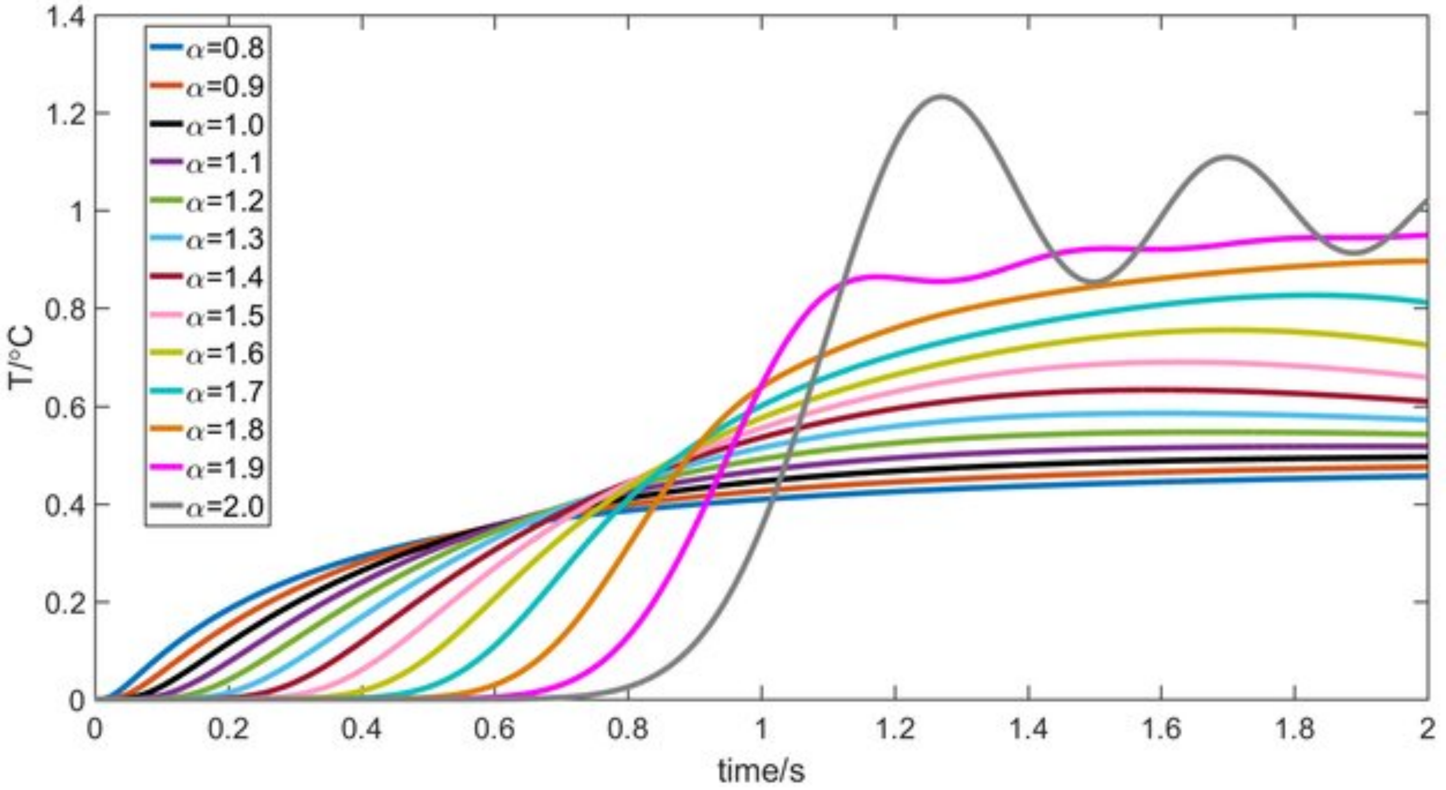
Fractional derivatives bridge the gap between differentiation and integration.

Applications

- Showcase some common fractional differential equations:

  - Fractional order heat equation:

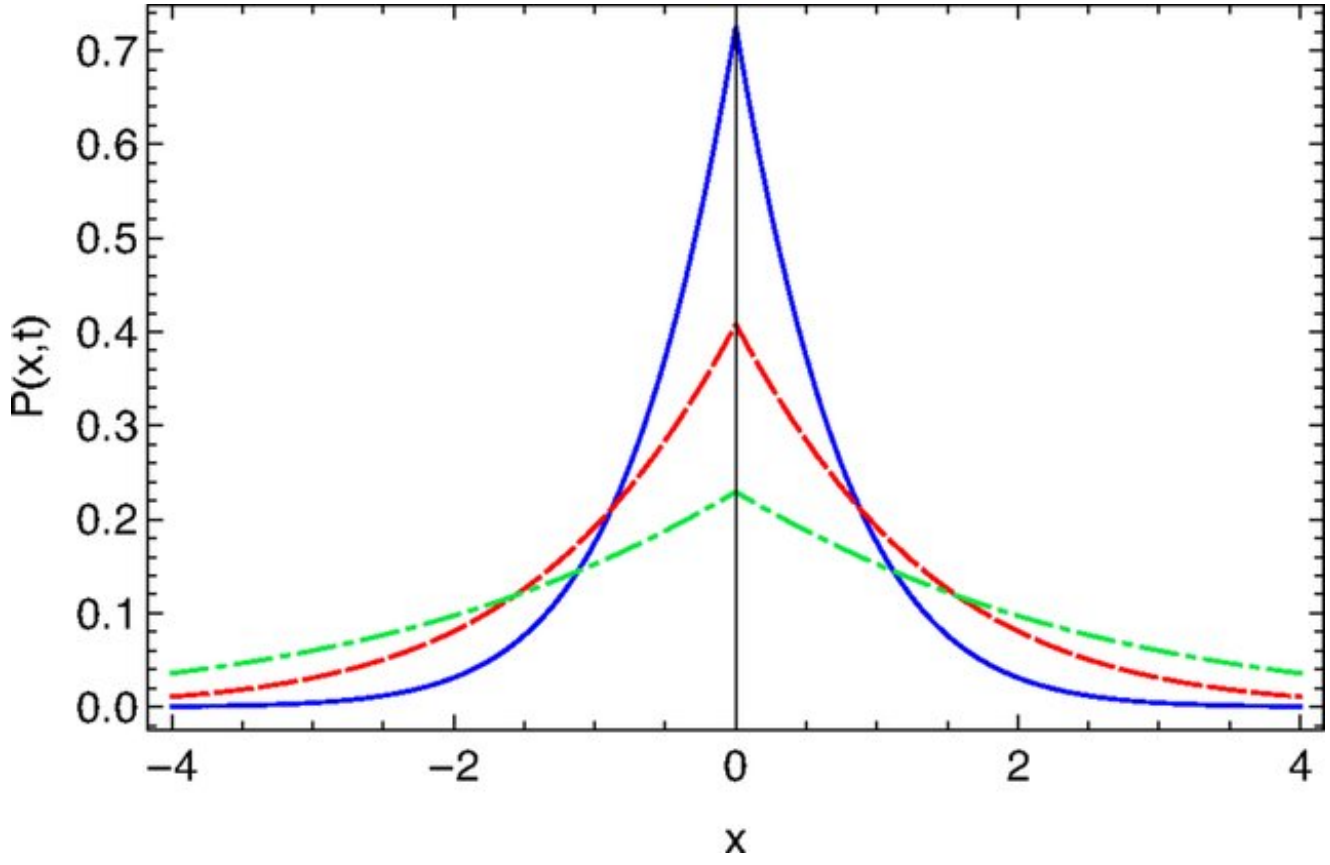
$$D_t^{\alpha} u(x,t) = K \frac{\partial^2 u(x,t)}{\partial x^2}$$



Applications

- Fractional order diffusion equation:

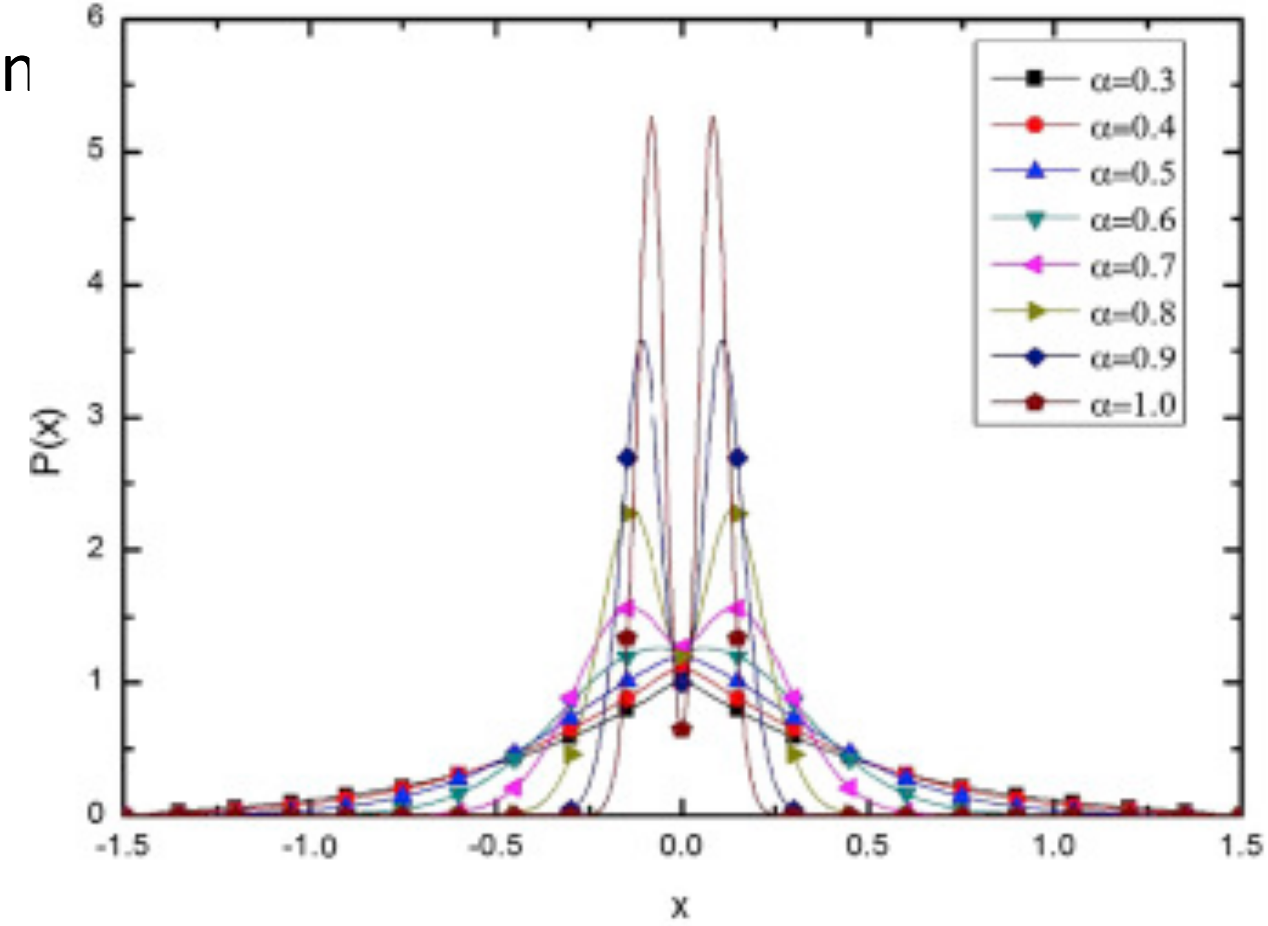
$$D_t^{\alpha} u(x,t) = K \frac{\partial^2 u(x,t)}{\partial x^2} - \lambda u(x,t)$$



Applications

- Fractional order wave equation

$$D_t^{\alpha} u(x,t) = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}$$



Applications

- **Image Processing** (for example reconstructing a degraded image)
- **Viscoelastic materials.**
- **Fluid flow.** (for solving fractional model of Navier Stokes eq. Arising in unsteady of viscous fluid)
- **Finiancial Modelling** (like Fractional Black –Scholes equations arising in financial markets)
- **Dynamics of earthquakes.**
- **Geology**.
- **Computaional Biology** (Such as time –fractional biological population models).
- **Signal processing.**
- **Chaotic dynamics.**
- **Medicine** (like Covid -19 recently)

Solving Fractional Differential Equations

Brief overview of solution techniques:

- Laplace transform method:

  - Convert the fractional differential equation to an algebraic equation in the Laplace domain.

  - Solve for the Laplace-transformed variable and then apply the inverse Laplace transform.

- Grünwald-Letnikov method:

  - Approximate the fractional derivative using finite differences.

  - Convert the fractional differential equation to a system of algebraic equations and solve numerically.

Solving Fractional Differential Equations

- Caputo's definition:

  - Convert the fractional differential equation to an integer-order differential equation using the Caputo fractional derivative definition.

  - Apply traditional techniques to solve the resulting integer-order differential equation.

- Mention numerical methods for approximate solutions:

  - Fractional Adams-Bashforth-Moulton method

  - Fractional Finite Difference method

  - Fractional Runge-Kutta method

Solving Fractional Differential Equations

- Employ some of the orthogonal polynomials in the numerical methods to solve fractional differential equations

  - investigate the operational matrices of fractional order (of integration or derivation)

  - based on these procedure the fractional differential equations converts to a system of algebraic equations can be solved numerically.

Thank you for listening